Daniel L. Burnstein

The Physics of Mathematical Practices and Infinities

Let me say at the outset that I am not happy with this state of affairs in physical theory. The mathematical continuum has always seemed to me to contain many features which are really very foreign to physics. […] If one is to accept the physical reality of the continuum, then one must accept that there are as many points in a volume of diameter 10^{13} cm or 10^{33} cm or 10^{1000} cm as there are in the entire universe. Indeed, one must accept the existence of more points than there are rational numbers between any two points in space no matter how close together they may be. (And we have seen that quantum theory cannot really eliminate this problem, since it brings in its own complex continuum.)

Roger Penrose, On the Nature of Quantum-Geometry

Incompleteness of purely mathematical proofs of infinities

Mathematics is a human activity. As such, the practice of mathematics is bound by the laws of physics. However, we act as if it operated outside of these laws.

For example, we learn early on in school that the set of natural numbers, the set of rational numbers, or that of irrational numbers all contain an infinite number of elements. We learn that the number line is a continuum composed of an infinite number of infinitesimal points, each of which corresponding to a real number. And to satisfy our inquisitive minds, our teachers provided irrefutable mathematical proofs of the existence of these (and other) infinities.

Consider the set of natural numbers $\mathbb{N}$. We know that $\forall x \in \mathbb{N}; x+1 \in \mathbb{N}$. That is, however large we chose x to be, we can always obtain a larger number by simply adding 1 to it. Since there is no upper limit to the size of x, the set of all natural numbers is thought to be limitless, hence infinite.

A similar reasoning is used to prove that the number line is continuous. However close two arbitrarily chosen points may be, we can always find an infinite number of points between them. There is an infinite number of points between even infinitesimally close points.

The above examples illustrate some of the irrefutable and definitive mathematical proofs of the existence of infinities. These proofs are so fundamental to our understanding of mathematics that we rarely question their validity.

But as we will show below, these purely mathematical proofs of infinities proofs are incomplete. They ignore the essential fact that mathematical objects, procedures, relations, concepts, etc., all are products of physical processes. And, as abstract as the mathematics may be, these processes require physical resources, energy to be brought into existence.

Even the constructivist argument according to which the existence of a number is proven only when it is constructed is paradoxical since it assumes that numbers have an objective existence prior to its constructions and ignores the physicality of such construction.

The fact is, you don't construct something that already exists, therefore constructing a number does not prove its existence, what it does is bring it into existence using processes are physical. The practice of mathematics is unquestionably physical.

Mathematics is practiced without concern about footing the bill for the material and energy resources it requires. But mathematical activities happen in physical reality and as such are not exempt from obeying the laws of physics. Every mathematical operation, the simplest addition or even the transcription of a result uses energy and consequently increases the universe's entropy. Even if there were an infinite amount of energy available for calculations, the number of operations will always be finite at any point in time.

When considering the physical aspect of mathematical proofs, we inevitably arrive at a counter proof of the existence of infinities.

Take proof that shows the set of natural numbers is infinite. We find that we cannot infinitely repeat the recursive operation implied in the mathematical proof. Each iteration requires energy and matter, each of the processes a sequence of causally linked states (the energy and steps of calculations necessary to construct them. And as any physical process, they rely on finite resources.

So, you can ultimately have a computer the size of the universe, with components the size the most fundamental particles, running at the optimal speed for the entire existence of the universe, exhausting all energy, and that still would be insufficient to construct an infinite set. The same reasoning applies to the continuum of the number line and to irrational numbers.

The ineluctable conclusion is that neither infinite sets nor continuums exists.

One may argue that there are infinite possibilities from which to chose when constructing a number, but possibilities are not actualities, and the act of choice is a process that is also limited by the laws of physics and physical resources.

The non-existence of infinities or continuums has implications for both mathematics and physics. We'll examine a some of them here.

Some Obvious Mathematical Implications

Numbers are constructed hence must have a finite number of decimals, this implies all numbers can be expressed as a ratio of two natural numbers, which must then include irrational numbers.

The number of decimals is then a function of the number of iterations of a computational procedure. Then $\sqrt{2}$ is not a number in itself, but represents a construction mode which may more appropriately be written in the form $\underset{i}{\sqrt{2}}$, where i is the number of iteration of the square root construction mode applied to 2.

The proof that $\frac{1}{3} \neq 0.33333...$

Taking into account that we apply i number of iterations of the division by three to the number 1, we find

$x = 0.33333...\underset{i}{3}$ and $10x = 3.33333...\underset{i-1}{3}$

hence

$$9x = 3.33333\ldots\underset{i-1}{3} - 0.33333\ldots\underset{i}{3} = 2.9999\ldots7$$

$$x = \frac{2.99999\ldots7}{9} \neq \frac{3}{9}\, or\, \frac{1}{3}$$

Inexistence of Irrational Numbers

By the same reasoning, we find that all constructed numbers must have a finite number of decimals, hence can be expressed as the ratio of two integer numbers, hence are rational.

It would also be interesting to reinterpret theorems in terms of modes of construction.

Take for instance Fermat's last theorem which states there exist no positive integers x, y, z and integer n such that for $n \geq 3$ the equation $x^n + y^n = z^n$ is satisfied. We can rewrite the equation as $\underset{i}{\sqrt[n]{x^n + y^n}} = z$ where the left side of the equation is the construction mode and the right side the result of the application of the construction after i number of iterations of the construction mode (the algorithm being defined as the set of operations that extends the decimal sequence).

Modes of construction of number generate rational numbers.

No Mathematical Geometrical Continuum

The non-existence of a continuous number line implies the non-existence continuous surfaces or volumes, or that of any continuous space regardless of dimensions.

Physical Infinities?

The notions of continuum and continuous mathematics have shaped our understanding of the nature of space.

The space continuum is physically also impossible for the reason stated by Sir. Roger Penrose, but also because it violates the laws of conservation of space and energy.

Nothing supports the assumption of space continuum. Virtually all our physics theories implicitly assume the space is a continuum despite all the problems, the unphysical implications, that this assumption causes (see John. C. Baez excellent summary of problems and paradoxes that arise in physics from the assumption of the existence of the continuum [1]).

We chose to believe in the continuum despite all the paradoxes and problems such belief cause and even though singularities and infinities do not even arise if is instead we worked from an assumption of space discreteness.

[1] See Struggles with the Continuum by John C. Baez arXiv 1609.01421

The existence of discrete unit of distance and discrete unit of space implies that all quantities are discrete. We already showed that there are no infinities in mathematics. There are no fractional lengths, surfaces, or volumes if space is fundamentally discrete.

A mathematical operation such as division of a length is therefore a division of an integer number of discrete units. That the length expressed in discrete units is not divisible by a certain number then simply means that the operation will produce quotient and remainder, but never a non integer result.

If the measurement is expressed in anything other than discrete units, then whatever conventional measuring unit is used in physics, we must always keep in mind when doing any sort of calculation that it corresponds to an integer number of fundamental discrete units.

There are no zeros in nature.

The number zero is one of the greatest inventions of mathematics. It would be very difficult to do any calculation without it. But zero is not like other numbers that we use to express quantities. Rather, by itself, it is a symbol to express the absence of quantity. Things physical exist, absent things do not. They are not physical. They can't be measured, weighed, or interacted with. They don't exist.

Used in conjunction with other numerical symbols, zeros are place holders allowing us to assign orders or magnitude to numbers. When we writ the number 10, or 100, or 1010, the zero are place holder that conventionally allows us to express the order of magnitude of the non-zero digits. Use in such ways, the resulting value of the number is always non-zero.

We may also argue that the non-existence of infinities and non-existence of zero in nature make division by zero inconsistent with the laws of physics and by extension, mathematically inconsistent since it also implies an infinity, that is, an infinite number of possible answers.

In discrete space, the distance between any two objects can never be zero because no objects can be separated by a distance smaller the fundamental unit of distance, hence gravity between any objects cannot become infinite, neither can there be objects of infinite density or any other kind of singularities.

The limit at which the result of a calculation becomes unphysical is when they are numerically smaller than the fundamental discrete unit of distance, or the fundamental unit of matter or momentum, both of which direct consequences of space discreteness.